Thesis topics

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1 BSc theses

1. **The Sylow theorems and classification of finite nonabelian groups.** The Lagrange theorem states that the order of a subgroup divides the order of the group. The converse is not necessarily true: There isn't a subgroup for every divisor of the order of a group. A partial converse result is provided by Sylow theorems. The Sylow theorems provide a powerful tool for characterizing finite nonabelian groups. The first goal of this topic is to state and prove the Sylow theorems. The second goal of this thesis is to classify all finite groups up to isomorphism. This project is from [Jud20, Chapter 15] where it is suggested to classify groups up to order 60, but the size can vary depending how quickly the thesis progresses.

   References: The main reference for the topic is [Jud20, Chapter 15]. It would be good to include also about the interesting recent history of classifying groups in the background chapter.

   Field: This topic is in the field of Abstract Algebra with the focus on group theory.

2. **Introduction to rigidity theory.** The main goal of rigidity theory is to determine whether there exists a unique configuration of n points in $\mathbb{R}^d$ up to rigid motions with a fixed partial set of pairwise distances between the points. The first goal of thesis is to characterize rigid motions, for example following [Jud20, Chapter 12]. The next step is to introduce the notions of infinitesimal, local and global rigidity, and state and prove some of the relations between the different types of rigidity. Global rigidity corresponds to the uniqueness of a configuration, but it is difficult to study. Especially infinitesimal rigidity provides an effective tool to obtain partial results.

   Reference: A book on rigidity theory (to be specified). See also [AR78, Rot81].

   Field: The rigidity theory is mainly a subfield of geometry and graph theory, but there are connections to abstract algebra (rigid motions) and linear algebra.

3. **Classification of the 17 wallpaper groups.** Wallpaper patterns are repeating patterns on the plane, for example as on a wallpaper. Mathematically, wallpaper patterns are characterized by lattices. Given two linearly independent vectors $x, y \in \mathbb{R}^2$, a lattice consists of all the integer combinations of x and y. Lattices are characterized by their symmetry groups, which are groups that map lattices to themselves. These symmetry groups are called wallpaper groups. The goal of this is to prove that there are precisely 17 different wallpaper groups.

   References: The starting point for this topic is [Jud20, Chapter 12], where further references for more detailed proofs can be found.

   Field: This topic is in the field of abstract algebra and mainly uses tools from group theory.

4. A chapter in geometric deep learning book [BBCV21], for example more generally on geometric deep learning or on group-equivariant CNNs.

2 MSc theses

1. **A fast algorithm for implicitization.** Implicitization is the problem of computing the kernel of a ring homomorphism

$$\begin{aligned} \phi : R = \mathbb{C}[x_1, \dots, x_n] &\to S = \mathbb{C}[t_1, \dots, t_m], \\ x_i &\mapsto \phi_i(x_i). \end{aligned}$$

   The standard tool for implicitization is via Gröbner basis computations, howeever, in practice it is often very slow and limited to small cases. Cummings and Hollering recently suggested a faster algorithm for implicitization [CH23]. The first goal of this topic is to describe the theory and algorithm for implicitization introduced in this paper. The second goal of this topic is to apply the algorithm to see if it allowes to compute the ideal for the five node binary Hidden Markov model. The ideal for the four-node binary Hidden Markov model was found in [Cri12]. The third goal is to try to compute the ideal for the five node binary Hidden Markov model using the `msolve` algorithm that is one of the best software for Gröbner basis computations, and also explain the theory behind `msolve`.

   References: The main reference for the theory is [CH23] and for the application is [Cri12]. It would be good also to briefly describe the widely used Gröbner bases approach for implicization, which can be found for example in [CLOS97].

   Field: This topic is in the fields of computational algebraic geometry, symbolic computation and commutative algebra. Part of the thesis is computational and requires doing experiments in `Macaulay2`, `OSCAR` and/or `msolve`.

2. Let $A \in \mathbb{Z}_{\geq 0}^{k \times n}$ be an integer matrix with columns $a_1, \ldots, a_n$. We consider the parametrization map

$$\phi^A : \mathbb{C}^k \to \mathbb{C}^n, \theta \mapsto (\theta^{a_1}, \ldots, \theta^{a_n}).$$

The toric variety X_A associated to a matrix $A \in \mathbb{Z}_{\geq 0}^{k \times n}$ is the Zariski closure of the image $\phi^A(\mathbb{C}^k)$. The log-linear model associated to a matrix $A \in \mathbb{Z}_{\geq 0}^{k \times n}$ is the set

$$\mathcal{M}_A = X_A \cap \Delta_{n-1},$$

where Δ_{n-1} is the $n-1$ dimensional probability simplex. Log-linear models are statistical models with nice properties, see also [Sul18, Chapter 6.2].

Let $\mathcal{M} \subseteq \Delta_{n-1}$ be a statistical model, $E \subseteq n$ a subset corresponding to observed entries and and $\pi_e : \mathbb{R}^{[n]} \to \mathbb{R}^E$ a projection map. A point $p_E \in \mathbb{R}^E$ is called partial observation and $p \in \pi_E^{-1}(p_E)$ a completion of p_E to model $\mathcal{M}$. Cay, Recke and Yahl study completion to discrete probability distributions in log-linear models in [CRY23].

Instead of toric varieties, one can also consider scaled toric varieties. Given $A \in \mathbb{Z}_{\geq 0}^{k \times n}$ and $c \in (\mathbb{C}^*)^n$, the scaled toric variety is $X_{A,c}$ is the Zariski closure of the map

$$\phi^{A,c} : \mathbb{C}^k \to \mathbb{C}^n, \theta \mapsto (c_1\theta^{a_1}, \ldots, c_n\theta^{a_n}).$$

Scaled log-linear models can be defined similarly as above. The goal of this MSc thesis project is to study to what extent the results in [CRY23] extend to scaled log-linear models.

Field: This topic is in the field of algebraic statistics. The main tools needed to study this topic come from toric geometry, which is a subfield of algebraic geometry that has close connections to polyhedral geometry. Only minimal knowledge of statistics is needed. One should have some background in (computational) algebraic geometry or toric geometry.

References

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[Cri12] Andrew J Critch. Binary hidden markov models and varieties. *arXiv preprint arXiv:1206.0500*, 2012.

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