PSD RANK AND NESTED SPECTRAHEDRA

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DEFINITION

- $\mathcal{S}_+^k$: the cone of $k \times k$ real symmetric positive semidefinite matrices
- $\mathcal{S}^k$: the vector space of real symmetric matrices equipped with the trace inner product

$$\langle A, B\rangle = \text{trace}(AB) = \sum_{1 \le i,j \le k} A_{ij}B_{ij}.$$

Definition 1 *The positive semidefinite (psd) rank of* $M \in \mathbb{R}_{\ge 0}^{m\times n}$ *is the smallest* k *such that there exist* $A_1, \dots, A_m, B_1, \dots, B_n \in \mathcal{S}_+^k$ *such that* $M_{ij} = \langle A_i, B_j\rangle$.

EXAMPLE

The matrix

$$M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

admits a psd factorization of size 2:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} A_3 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} B_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

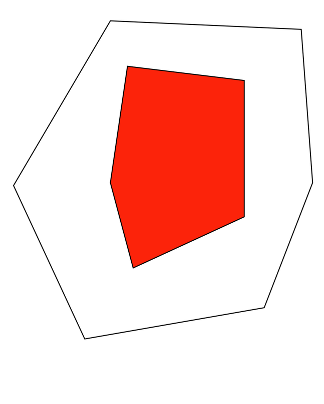
GEOMETRIC INTERPRETATION

Rank: the smallest $r \in \mathbb{N}$ such that $M = AB$ where A has r columns and B has r rows.

$$\begin{bmatrix} 157 & 181 & 191 \\ 374 & 432 & 456 \\ 734 & 852 & 900 \\ 1094 & 1272 & 1344 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \\ 11 & 13 \\ 17 & 19 \end{bmatrix} \begin{bmatrix} 23 & 29 & 31 \\ 37 & 41 & 43 \end{bmatrix}$$

If $M \in \mathbb{R}_{\ge 0}^{m\times n}$, then such a factorization can be pictured with a pair of nested polytopes in $\mathbb{RP}^{r-1}$:

- outer polytope: $Q = \{x \in \mathbb{R}^r : Ax \ge 0\}$
- inner polytope: $P = \text{conv}(B)$



Theorem 1 (Gouveia, Robinson, Thomas) *The matrix* M *has size* k *psd factorization if and only if there exists an affine subspace* $\mathcal{L}$ *of* S^k *and a linear map* π *such that* $P \subseteq \pi(S_+^k \cap \mathcal{L}) \subseteq Q$.

MOTIVATION

Let $P \subset \mathbb{R}^n$ be a polytope and assume we want to minimize a linear function ℓ over P, i.e.

$$\min \quad \ell(x)$$

$$\text{s.t.} \quad x \in P.$$

If P admits a representation of the form

$$P = \pi(\mathcal{S}_+^k \cap \mathcal{L})$$

where $\mathcal{L} \subset \mathcal{S}^k$ is an affine subspace and π a linear map, then

$$\min_{x\in P} \ell(x) = \min_{y \in \mathcal{S}_+^k \cap \mathcal{L}} \ell \circ \pi(y),$$

which is a semidefinite program.

EXAMPLE: MOTIVATION

Let $P = Q = [-1, 1]^2$. Then, the slack matrix of P is

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

It has psd rank 3.

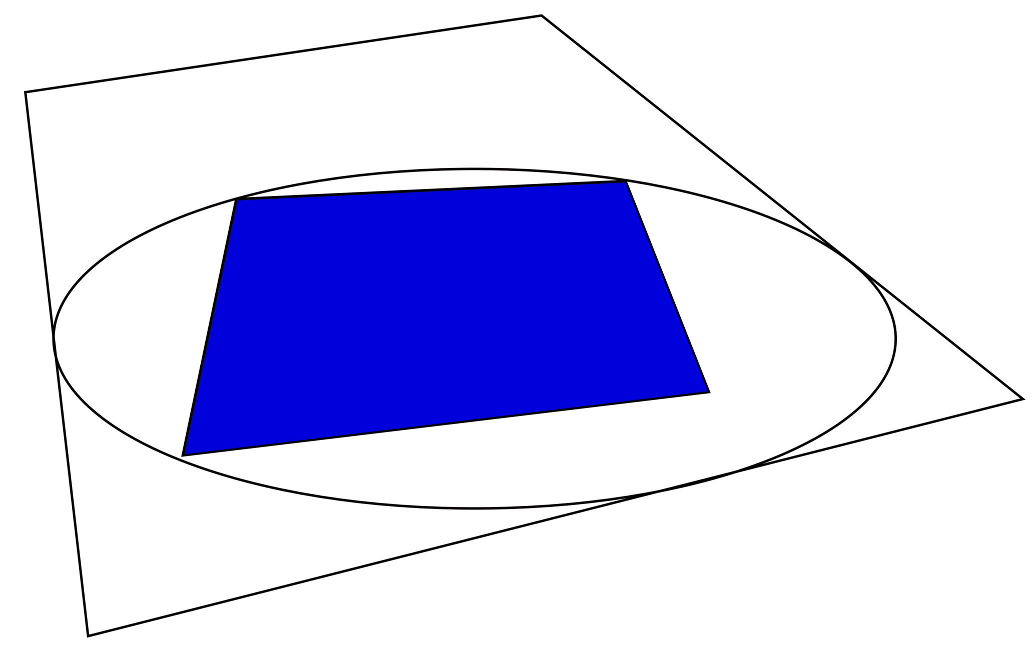
Consider

$$T = \left\{ (x, y, z) \in \mathbb{R}^3 : \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0 \right\}.$$

P is the projection of T onto the (x, y) coordinates.

PSD RANK 2

Let $\mathcal{M}_{r,k}^{p\times q}$ denote the semialgebraic set of matrices of rank at most r and psd rank at most k.



Theorem 2 *We describe the topological and algebraic boundaries of* $\mathcal{M}_{3,2}^{p\times q}$.

a. *A matrix* $M \in \mathcal{M}_{3,2}^{p\times q}$ *lies on the topological boundary* $\partial\mathcal{M}_{3,2}^{p\times q}$ *if and only if* $M_{ij} = 0$ *for some* i, j, *or each ellipse that fits between the two polygons* P *and* Q *contains at least 3 vertices of the inner polygon* P *and is tangent to at least 3 sides of the outer polygon* Q.

b. *A matrix* $M \in \overline{\mathcal{M}_{3,2}^{p\times q}} = \mathcal{V}_3$ *lies on the algebraic boundary* $\partial\mathcal{M}_{3,2}^{p\times q}$ *if and only if* $M_{ij} = 0$ *for some* i, j *or there exists an ellipse that contains at least three vertices of* P *and is tangent to at least three edges of* Q.

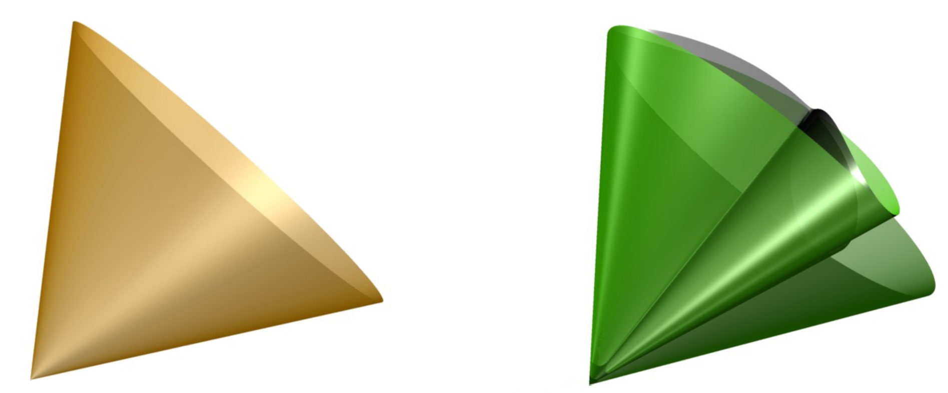
c. *The algebraic boundary of* $\mathcal{M}_{3,2}^{p\times q}$ *is the union of* $\binom{p}{3}\binom{q}{3} + pq$ *irreducible components. Besides the* pq *components* $M_{ij} = 0$, *there are* $\binom{p}{3}\binom{q}{3}$ *components each of which is defined by the* 4×4 *minors of* M *and one additional polynomial equation with 1035 terms homogeneous of degree 24 in the entries of* M *and homogeneous of degree 8 in each row and each column of a* 3×3 *submatrix of* M.

EXAMPLE: CIRCULANT MATRICES

3×3 circulant matrices:

$$\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$$

Circulant matrices of psd rank at most 2 and the boundary polynomial:



HIGHER PSD RANK

Conjecture 1 *A matrix* M *is on the boundary* $\partial\mathcal{M}_{k+1,k}^{p\times q}$ *if and only if for all spectrahedral shadows* C *such that* $P \subseteq C \subseteq Q$, *the shadow* C *contains* $k+1$ *vertices of* P *at rank one loci and touches* $k+1$ *facets of* Q *at rank* $k-1$ *loci.*

Conjecture 2 *A matrix* $M \in \mathcal{M}_{k+1,k}^{p\times q}$ *lies on the boundary* $\partial\mathcal{M}_{k+1,k}^{p\times q}$ *if and only if for every psd factorization* $M_{ij} = \langle A_i, B_j\rangle$ *with* $A_i, B_j \in \mathcal{S}_+^k$, *there exist* $1 \le i_1 < \dots < i_{k+1} \le p$ *and* $1 \le j_1 < \dots < j_{k+1} \le q$ *such that*

$$\text{rank}(A_{i_1}) = \dots = \text{rank}(A_{i_{k+1}}) =$$
$$\text{rank}(B_{j_1}) = \dots = \text{rank}(B_{j_{k+1}}) = 1.$$

REFERENCES

- Positive semidefinite rank. Hamza Fawzi, João Gouveia, Pablo A. Parrilo, Richard Z. Robinson, Rekha R. Thomas. arXiv:1407.4095.
- Positive semidefinite rank and nested spectrahedra. Kaie Kubjas, Elina Robeva, Richard Z. Robinson. arXiv:1512.08766.