GROUP-BASED MODELS AND BERENSTEIN-ZELEVINSKY TRIANGLES

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PHYLOGENETIC MODELS

- General group-based models are algebraic varieties associated to a tree T and an Abelian group G coming from evolutionary biology.
- They are projective toric varieties, hence can be described by lattice polytopes and affine semigroups.

JUKES-CANTOR BINARY MODEL

- The Jukes-Cantor binary model is the simplest general group-based model corresponding to $G = \mathbb{Z}_2$.
- Let T be a tree, E its set of edges and I its set of inner vertices.

Definition. The **polytope** $P_{\mathbb{Z}_2,T}$ and the **semigroup** $S_{\mathbb{Z}_2,T}$ of the Jukes-Cantor binary model associated to a tree T are defined by

$$P_{\mathbb{Z}_2,T} = \operatorname{conv}(a \in \{0,1\}^E : \sum_{e \ni v} a_e \in 2\mathbb{Z} \text{ for every } v \in I),$$

$$S_{\mathbb{Z}_2,T} = \mathbb{N}[\operatorname{vert}(P_{T,\mathbb{Z}_2}) \times \{1\}]$$

The Hilbert polynomial of the Jukes-Cantor binary model has a beautiful property:

Theorem (Buczyńska and Wiśniewski [BW07]). *Let T_1 and T_2 be 3-valent trees with the same number of leaves. Then*

$$\operatorname{hilb}(S_{\mathbb{Z}_2,T_1}, t) = \operatorname{hilb}(S_{\mathbb{Z}_2,T_2}, t).$$

Example.

$$\operatorname{hilb}(S_{\mathbb{Z}_2,\text{snowflake}}, t) = \operatorname{hilb}(S_{\mathbb{Z}_2,\text{3-caterpillar}}, t)$$

snowflake 3-caterpillar

GENERAL GROUP-BASED MODELS

- Let G be an Abelian group and T be a tree.
- Label every edge $e \in E$ with an element $g_e \in G$ such that the sum of labels around each inner vertex is 0.

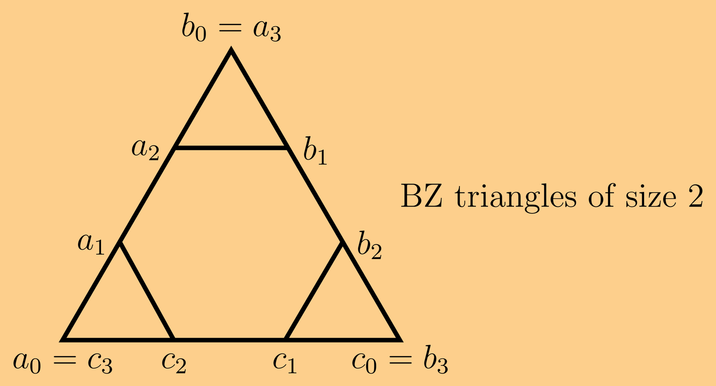
Definition. The **polytope** $P_{G,T}$ is defined by

$$\operatorname{conv}(x \in \mathbb{Z}^{E \times G} | a^x = \prod_{e \in E(T)} a^{(e)}_{g_e} \text{ for a labeling } (g_e)_{e \in E}).$$

INVARIANCE OF THE HILBERT POLYNOMIAL

- In general, the Hilbert polynomial of a general group-based model does depend on the shape of the tree (Kubjas [Kub10], Michałek and Donten-Bury [MDB10]).
- However, Sturmfels and Xu [SX10], Manon [Man09] related the Jukes-Cantor binary model to $\mathrm{sl}_2(\mathbb{C})$ conformal block algebras: Semigroups $S_{\mathbb{Z}_2,T}$ can be constructed from BZ triangles of size 1.
- Similarly, we can construct semigroups from BZ triangles of size 2 whose Hilbert polynomials are independent of the tree topology.

BERENSTEIN-ZELEVINSKY TRIANGLES



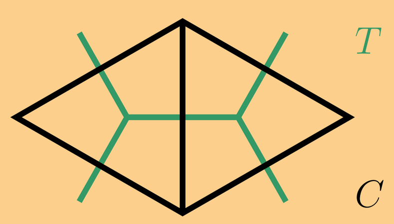
- label the vertices of the graph with non-negative integers such that opposite edges of the hexagon have equal sums of labels
- all possible labelings form the **semigroup** K_2 of BZ triangles of size 2, which is generated by 8 basic BZ triangles $\Delta_1, \ldots, \Delta_8$
- the **semigroup** K_2^* of graded BZ triangles of size 2 is

$$\mathbb{N}[\{\Delta_i : i \in \{1, \ldots, 8\}\} \times \{1\}, \{0\} \times \{1\}]$$

- for the construction we use the **projection** $\mathrm{pr}\colon K_2 \to \mathbb{R}^6$

$$\mathrm{pr}(a, b, c) = (a_0 + a_1, a_2 + a_3, b_0 + b_1, b_2 + b_3, c_0 + c_1, c_2 + c_3)$$

- let T be a 3-valent tree and C a triangle complex dual to T



- assign a graded BZ triangle to every triangle in C such that
  - all BZ triangles have the same degree
  - any two BZ triangles sharing an edge have equal projections on the edge
- all possible assignments together with the degree form the **semigroup** $K_{2,T}$

SEMIGROUPS ARE RELATED

The semigroups $\mathbb{N}[\operatorname{vert}(P_{T,\mathbb{Z}_{r+1}}) \times \{1\}]$ and K_r are related by the projection.

Lemma. *Let T be the 3-valent 3-leaf tree. Then*

$$pr(K_r) \supseteq \mathbb{N}[vert(P_{T,\mathbb{Z}_{r+1}}) \times \{1\}].$$

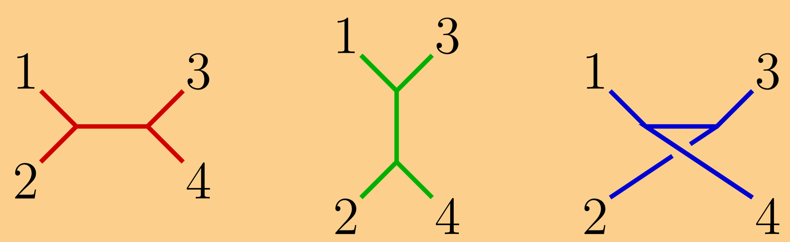
In particular, for $r \in \{1, 2\}$ the equality holds.

HILBERT POLYNOMIAL OF $\mathrm{SL}_3(\mathbb{C})$

Theorem. *Let T_1 and T_2 be 3-valent trees with the same number of leaves. Then*

$$h(K_{2,T_1}, t) = h(K_{2,T_2}, t).$$

Proof.
- there are 3 non-isomorphic labelings of the 3-valent 4-leaf tree T



- each labeling induces a multigrading of $K_{2,T}$ defined by projections
- using `Macaulay2` we show that the Hilbert series of $K_{2,T}$ with respect to different multigradings are equal
- by using toric fiber products, this implies that the Hilbert series does not depend on the topology of the tree

□

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